LS-132
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November 1988

ANALYSIS OF THE WALKINSHAW DIFFERENCE RESONANCE

In preparation for the Aladdin experiments, I will give an analytic treatment of the Walkinshaw difference resonance. The treatment nearly parallels that in LS-131 for the third-integral resonance.

1. Analysis of the Resonance

The Hamiltonian in the neighborhood of the Walkinshaw resonance

$$\nu_x - 2\nu_y = m \tag{1.1}$$

can be written in terms of angle-action variables in the form

$$\begin{aligned} h = {} & \nu_x J_x + \nu_y J_y + S(2J_x)^{1/2}(2J_y)\sin(\gamma_x - 2\gamma_y - m\theta + \zeta) \\ & + aJ_x^2 + 2bJ_xJ_y + cJ_y^2 \quad . \end{aligned} \tag{1.2}$$

We first transform to resonant coordinates via the generating function

$$F(J_1,J_2,\gamma_x,\gamma_y,\theta) = J_1(\gamma_x - 2\gamma_y - m\theta + \zeta) + J_2\gamma_y \quad , \tag{1.3}$$

which gives

$$\gamma_1 = \gamma_x - 2\gamma_y - m\theta + \zeta \quad , \quad \gamma_2 = \gamma_y \quad , \tag{1.4}$$

$$J_x = J_1 \quad , \quad J_y = J_2 - 2J_1 \quad , \tag{1.5}$$

$$J_1 = J_x \quad , \quad J_2 = J_y + 2J_x \quad . \tag{1.6}$$

The resonant hamiltonian is

$$h_w = \varepsilon J_1+\nu_y J_2+2S(2J_1)^{1/2}(J_2-2J_1)\sin\gamma_1+aJ_1{}^2+2bJ_1(J_2-2J_1)$$
$$+c(J_2-2J_1)^2 \quad , \tag{1.7}$$

where

$$\varepsilon = \nu_x - 2\nu_y - m \quad . \tag{1.8}$$

We see that J_2 is a constant of the motion. Note also that the motion is required by Eq. (1.5) to lie within the circle $2J_1 < J_2$ in the J_1, γ_1 -phase plane.

In rectangular coordinates

$$Q = (2J_1)^{1/2}\sin\gamma_1 \quad , \quad P = (2J_1)^{1/2}\cos\gamma_1 \quad , \tag{1.9}$$

the hamiltonian is

$$h_w = \tfrac{1}{2}\varepsilon\ (P^2+Q^2) + 2SQ(J_2-P^2-Q^2)+ \tfrac{1}{4}\ a\ (P^2+Q^2)^2+b(P^2+Q^2)(J_2-P^2-Q^2)$$

$$+c(J_2-P^2-Q^2)^2 + \nu_y J_2 \quad . \tag{1.10}$$

For the value of h_w corresponding to the limiting circle $J_2 - 2J_1 = 0$, Eq. (1.10) factors into the product of two circles:

$$(J_2-P^2-Q^2)[2SQ+b(P^2+Q^2)+c(J_2-P^2-Q^2)- \tfrac{1}{2}\ \varepsilon - \tfrac{1}{4}\ aJ_2- \tfrac{1}{4}a(P^2+Q^2)] = 0 \quad . \tag{1.11}$$

The first factor is the limiting circle, and the second is the circle

$$P^2 + (Q + \frac{S}{A})^2 = \frac{B}{A} + \frac{S^2}{A^2} \quad , \tag{1.12}$$

where

$$A=b-c-\frac{1}{4}a \quad , \; B = \frac{1}{2}\varepsilon-(c-\frac{1}{4}a)J_2 \quad . \tag{1.13}$$

The circles intersect (if at all) in the points

$$Q_o=(B-AJ_2)/2S \quad , \quad P_o= \pm(J_2-Q_o{}^2)^{1/2} \quad . \tag{1.14}$$

There are two cases:

a) $|Q_o| > J_2{}^{1/2}$. Circles do not intersect. (See Fig. 1.)

b) $|Q_o| < J_a{}^{1/2}$. Circles intersect. (See Fig. 2.) The elliptic fixed points occur at the points.

$$Q_{10\pm} = \frac{\varepsilon}{12S} \{1\pm[1+ \frac{48S^2J_2}{\varepsilon^2}]^{1/2}\} \quad , \tag{1.15}$$

if we neglect the higher-order frequency shifting terms. The first-order correction for frequency-shift terms is

$$Q_{1\pm} = Q_{10\pm} \pm \frac{\varepsilon\Delta K}{[\varepsilon^2+48S^2J_2]^{1/2}} \quad , \tag{1.16}$$

where

$$\Delta K = (2b - 4c)\, J_2Q_{10\pm} + (a+4c - 4b)\, Q_{10\pm}{}^3 \quad . \tag{1.17}$$

It is convenient, both experimentally and theoretically to consider motions in which the y-amplitude is initially very small and the x-amplitude is finite. In that case, $J_2 = x_o^2$, where x_o is the initial x-amplitude. The conditions above become

a) (Fig. 1) $x_o < \left| [\varepsilon + (a-2b)x_o^2]/4S \right|$,

b) (Fig. 2) $x_o > \left| [\varepsilon + (a-2b)x_o^2]/4S \right|$. (1.18)

The term $(a-2b)x_o^2$ is a higher order correction due to amplitude-dependent tune shifts. From Eq. (1.4), we see that the condition that y remains small is that the phase point Q, P remains near the limiting circle. Therefore, in case a), there is no coupling. A small initial y-amplitude remains small. In case b), the phase point cannot remain near the limiting circle. Thus if x exceeds the threshold value given by Eq. (1.18), the x and y motions are coupled. the maximum y-amplitude occurs where the circular arc crosses the Q-axis, i.e., at $(\pm)Q_2$, where

$$Q_2 = \left[\frac{B}{A} + \frac{S^2}{A^2}\right]^{1/2} - \left|\frac{S}{A}\right| \quad , \tag{1.19}$$

and $(\pm)$ is the sign of S/A.

The maximum y-amplitude is

$$\begin{aligned} y_{max}^2 &= 2(x_o^2 - Q_2^2) \\ &= 2x_o^2 - \frac{\varepsilon^2}{8S} + \frac{b\varepsilon^3}{128S^4} + (c - \tfrac{1}{4}a)\frac{\varepsilon}{2S^2}\left(x_o^2 - \frac{\varepsilon^2}{64S^2}\right) + \ldots \quad , \end{aligned} \tag{1.20}$$

where the last line gives the first order correction due to amplitude-dependent tune shifts.

It is of interest to estimate the frequency of energy exchange between x and y motion in case b). Since the points Q_o, $\pm P_o$ are fixed points, the period for zero initial y-amplitude is infinite. For small initial y-amplitude, the phase point moves around the limiting circle in Fig. 2 until it arrives at the fixed point. It remains near the fixed point for a time, depending on the y-amplitude, and then traverses the circular arc up to the other fixed point. The y-motion therefore consists of a long time near zero amplitude, punctuated by periodic increases to a maximum given by Eq. (1.20) and subsequent fall to near zero amplitude. If we neglect the terms in a, b, and c, the circular arc becomes the straight line

$$Q = \frac{\varepsilon}{4S} \quad . \tag{1.21}$$

We can integrate the equation of motion for P along this line, to obtain

$$P = P_o \tanh (2P_o S\theta) \quad , \tag{1.22}$$

where P_o is given by Eq. (1.14):

$$P_o^{\ 2} = x_o^{\ 2} - \frac{\varepsilon^2}{16S^2} \quad . \tag{1.23}$$

The time scale for the pulse in y-amplitude is therefore

$$\frac{\Delta\theta}{2\pi} = \frac{1}{4\pi P_o S} \sim \frac{1}{4\pi x_o S} \text{ revolutions.} \tag{1.24}$$

We can calculate the frequency of motion about the fixed points $Q_{1\pm}$. The result, for $\varepsilon = 0$ and neglecting a, b, and c is

$$\nu_w = 4x_o S \text{ oscillations/revolution.} \tag{1.25}$$

The time scale is of the same order as that given by Eq. (1.24).

These time scales are shorter than radiation damping times, so it should be possible to see coupling phenomena. Radiation damping will damp the motion in Fig. 2 toward the fixed points, at the same time damping the value of J_2, which will move the system toward the non-resonant case, unless $\varepsilon = 0$.

2. Connection with the Real Ring

We follow the same analysis as in LS-131 through Eq. [2.7]. (I will use square bracket to indicate equation numbers from LS-131.) We substitute from Eq. [2.3] in Eq. [2.6], to obtain, in place of Eq. [2.8],

$$H_w = R^{-1}H_o(J_x,J_y) + \frac{B''\ell}{6B\rho}\,\delta(s-s_j)(2J_x)^{3/2}\beta_x^{3/2}\sin^3(\gamma_x-\psi_x)$$

$$-\frac{B''\ell}{2B\rho}\,\delta(s-s_j)\beta_x^{1/2}\beta_y(2J_x)^{1/2}(2J_y)\sin(\gamma_x-\psi_x)\sin^2(\gamma_y-\psi_y)\quad, \tag{2.1}$$

where

$$H_o(J_x,J_y) = \nu_xJ_x+\nu_yJ_y+aJ_x^2+2bJ_xJ_y+cJ_y^2\quad. \tag{2.2}$$

We now make the substitutions [2.9] and [2.10], with the result, in place of Eq. [2.11],

$$H_\theta = H_o(J_x,J_y) + \sum_{m=-\infty}^{\infty} S_3(2J_x)^{3/2}[-\sin(3\gamma_x-m\theta+ms_j/R-3\psi_{xj})$$

$$+3\sin(\gamma_x-m\theta+ms_j/R-\psi_{xj})]$$

$$-\sum_{m=-\infty}^{\infty} S_w(2J_x)^{1/2}(2J_y)[-\sin(\gamma_x+2\gamma_y-m\theta+ms_j/R-\psi_{xj}-2\psi_{yj})$$

$$-\sin(\gamma_x-2\gamma_y-m\theta+ms_j/R-\psi_{xj}+2\psi_{yj})+ 2\sin(\gamma_x-m\theta+ms_j/R-\psi_{xj})]\quad, \tag{2.3}$$

where S_3 is given by Eq. [2.12], and

$$S_w = \left(\frac{\beta_x^{1/2}\beta_y B''\ell}{16\,\pi\, B\rho}\right)_{s=s_j}\quad. \tag{2.4}$$

After dropping or transforming away the non-resonant terms, we are left with the Hamiltonian (1.2), with S given by Eq. (2.4), and

$$\zeta = ms_j/R-\psi_{xj}+2\psi_{yj}\quad. \tag{2.5}$$

3. Transforming the Non-Resonant Terms

The non-resonant terms in Eq. (2.3) can be transformed away by the method used in LS-131, Section 3. We introduce a generating function

$$F = \underline{J}_x \gamma_x + \underline{J}_y \gamma_y + (2\underline{J}_x)^{3/2} \Sigma_3 + (2\underline{J}_x)^{1/2} (2\underline{J}_y) \Sigma_w \quad , \tag{3.1}$$

where

$$\Sigma_3 = \sum_m [F_{m'3} \cos(3\gamma_x - m\theta + ms_j/R - 3\psi_{xj}) + F_{m'31} \cos(\gamma_x - m\theta + ms_j/R - \psi_{xj})] \quad ,$$

$$\begin{aligned} \Sigma_w = \sum_{m'} [&F_{m'+} \cos(\gamma_x + 2\gamma_y - m\theta + ms_j/R - \psi - 2\psi_{yj}) \\ &+ F_{m'-} \cos(\gamma_x - 2\gamma_y - m\theta + ms_j/R - \psi_{xj} + 2\psi_{yj}) \\ &+ F_{m'w1} \cos(\gamma_x - m\theta + ms_j/R - \psi_{xj})] \quad . \end{aligned} \tag{3.2}$$

The transformation equations for J are

$$J_x = \frac{\partial F}{\partial \gamma_x} = \underline{J}_x + (2\underline{J}_x)^{3/2} \frac{\partial \Sigma_3}{\partial \gamma_x} + (2\underline{J}_x)^{1/2} (2\underline{J}_y) \frac{\partial \Sigma_w}{\partial \gamma_x} \quad ,$$

$$J_y = \frac{\partial F}{\partial \gamma_y} = \underline{J}_y + (2\underline{J}_x)^{3/2} \frac{\partial \Sigma_3}{\partial \gamma_y} + (2\underline{J}_x)^{1/2} (2\underline{J}_x) \frac{\partial \Sigma_w}{\partial \gamma_y} \quad . \tag{3.3}$$

The third-order terms in $\underline{H}_\theta$ can be written as in Eq. [3.4]. They can be made to vanish by setting

$$F_{m'3} = \frac{-S_3}{3\nu_x - m'} \quad , \qquad F_{m'31} = \frac{3\ S_3}{\nu_x - m'} \quad ,$$

$$F_{m'+} = \frac{S_w}{\nu_x + 2\nu_y - m'} \quad , \qquad F_{m'-} = \frac{S_w}{\nu_x - 2\nu_y - m'} \quad , \qquad F_{m'w1} = \frac{-2\ S_w}{\nu_x - m'} \quad . \tag{3.4}$$

We do not want to transform away the resonant term, so we set $F_{m-} = 0$, for the value $m' = m$ corresponding to the resonance (1.1). (In Aladdin, $m = -7$.)

We can now calculate the θ-independent terms of fourth order in $\underline{H}_\theta$. The resulting corrections to the coefficients in Eq. (2.2) are

$$a_s = 6S_3^{\,2} \sum_{m'} \left[\frac{3}{m'-3\nu_x} + \frac{1}{m'-\nu_x}\right] \doteq 6S_3^{\,2}\left[-\frac{1}{\delta\nu_x} - \frac{3}{\delta(3\nu_x)} + 2\delta\nu_x + 6\delta(3\nu_x)\right] , \qquad (3.5)$$

$$b_s = 3S_3 \sum_{m'} (-3F_{m'w1}) - S_w \sum_{m'} [-2F_{m'31} - 4F_{m'w1} + 2F_{m'+} + 2F_{m'-}]$$

$$= \sum_{m'}{}' \left\{\frac{24S_3S_w - 8S_w^{\,2}}{\nu_x - m'} - \frac{2S_w^{\,2}}{\nu_x + 2\nu_y - m'} - \frac{2S_w^{\,2}}{\nu_x - 2\nu_y - m'}\right\}$$

$$\doteq (24S_3S_w - 8S_w^{\,2})\left[\frac{1}{\delta\nu_x} - 2\delta\nu_x\right] - 2S_w^{\,2}\left[\frac{1}{\delta(\nu_x+2\nu_y)} - 2\delta(\nu_x+2\nu_y) - 2\delta(\nu_x-2\nu_y)\right] \qquad (3.6)$$

$$c_s = -2S_w \sum_{m'} [-2F_{m'w'1} + F_{m'+} + F_{m'-}] = -12S_w^2 \sum_{m'}{}' \left[\frac{4}{\nu_x - m'} + \frac{1}{\nu_x + 2\nu_y - m'} + \frac{1}{\nu_x - 2\nu_y - m'}\right]$$

$$\doteq -2S_w^{\,2}\left[\frac{4}{\delta\nu_x} + \frac{1}{\delta(\nu_x+2\nu_y)} - 4\delta\nu_x - \delta(\nu_x+2\nu_y) - \delta(\nu_x-2\nu_y)\right] . \qquad (3.7)$$

Formula (3.5) agrees with [3.7], except that there is no missing resonant term. The term δz is again defined as $z-m$, where m is the nearest integer to z.

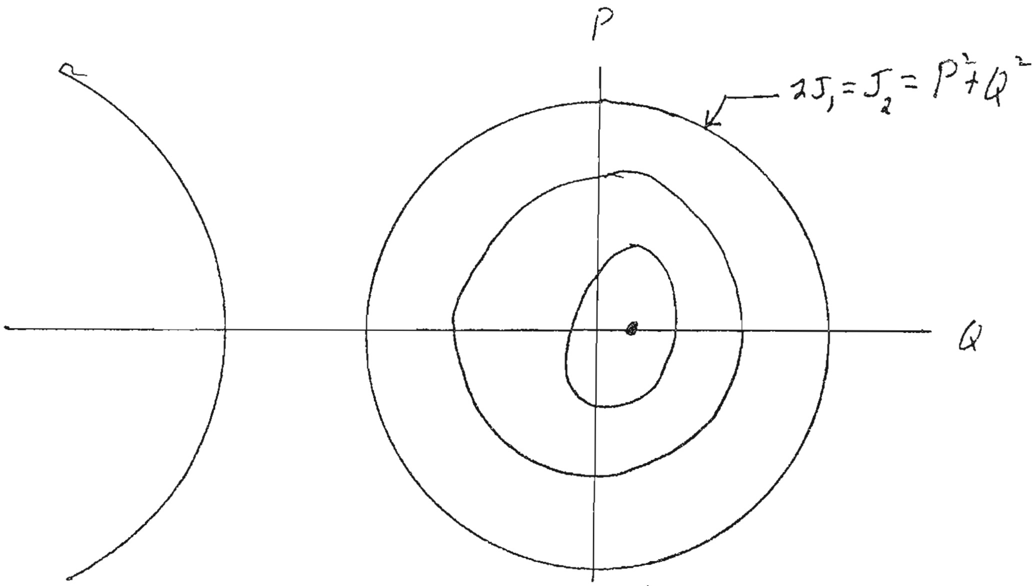

Fig. 1. Non-resonant case.

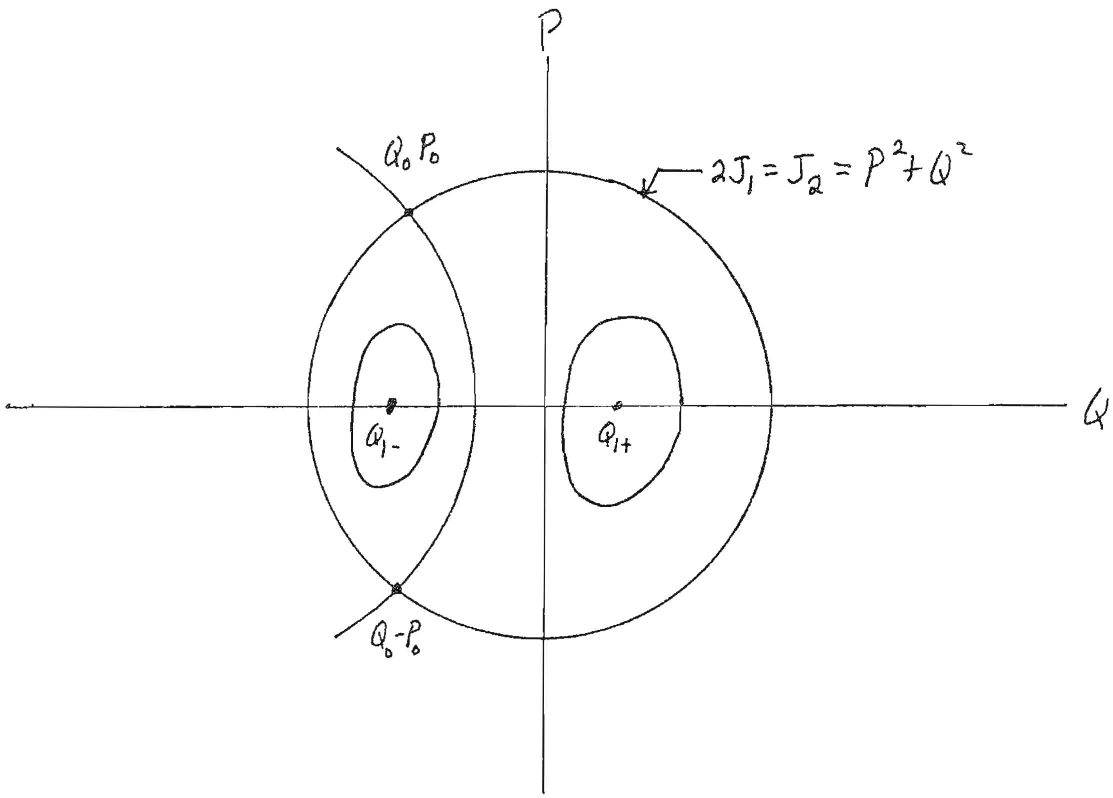

Fig. 2. Resonant Case